Math 122 / Problem Set 6 Solution

By Tien Anh Nguyen - thanks to Yiyi Deng's and Tagop Taminian's very good write ups for problems 1 and 10

Problem 1

Let $D_{10} = \{x^i y^j | x^{10} = y^2 = 1, xy = yx^9\}$.
a) Since $\frac{\#D_{10}}{\#H} = 10$, we expect there to be 10 cosets. Also since $x^5 y = yx^5$, direct calculation gives $1H = H, xH = x^6H = \{x, x^6\}, x^2H = x^7H = \{x2, x^7\}, x^3H = x^8H = \{x^3, x^8\}, x^4H = x^9H = \{x^4, x^9\}, yH = x^5yH = \{y, x^5y\}, xyH = x^6yH = \{xy, x^6y\}, x^2yH = x^7yH = \{x^2y, x^7y\}, x^3yH = x^8yH = \{x^3y, x^8y\}, x^4yH = x^9yH = \{x^4y, x^9y\}$.
b) Define $\phi : D_{10} \to D_5, \phi(x^i y^j) = \tilde{x}^i \tilde{y}^j$, where x, y generate D_{10} as above and $\tilde{x}, \tilde{y}$ generates D_5.
Check that this is a homomorphism: $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$. Indeed, let $a = x^i y^j, b = x^s y^t$,
then $\phi(x^i y^j x^s y^t) = \left[\begin{array}{l} \phi(x^{i+s} y^t) = \tilde{x}^{i+s} \tilde{y}^t \text{ if } j = 0 \\ \phi(x^{i-s} y^{1+t}) = \tilde{x}^{i-s} \tilde{y}^{1+t} \text{ if } j = 1 \end{array} \right.$
However, $\tilde{x}^{i-s}\tilde{y}^{1+t} = \tilde{x}^i \tilde{x}^{-s} \tilde{y} \tilde{y}^t = \tilde{x}^i \tilde{y} \tilde{x}^s \tilde{y}^t = \phi(x^i y^j)\phi(x^s y^t)$, and similarly, $\tilde{x}^{i+s}\tilde{y}^t = \tilde{x}^i \tilde{y}^0 \tilde{x}^s \tilde{y}^t = \phi(x^i)\phi(x^s y^t)$ when $j = 0$. In both cases, we have an equality, therefore our map is a homomorphism of groups.
Any element in D_5 has the natural preimage in D_{10}, by simply replacing $\tilde{x}$ and $\tilde{y}$ with x and y, hence the map is surjective. Since $\tilde{x}^5 = 1$, the kernel of $\phi = \{1, x^5\} = H$.
Hence by the First Isomorphism Theorem,$D_{10}/H \simeq D_5$.
c) We define $f : D_5 \times H \to D_{10}, f(\tilde{x}^i \tilde{y}^j, x^k) = x^i y^j x^k$ (multiplication in D_{10})
We first show that this is a homomorphism: if $(a, b) = (\tilde{x}^i \tilde{y}^j, x^k), (a', b') = (\tilde{x}^s \tilde{y}^t, x^u)$, (note that H is considered a subgroup of D_{10}, while D_5 is not.Then
$f((a, b) \cdot (a', b')) = f(\tilde{x}^i \tilde{y}^j x^s \tilde{y}^t, x^k x^u) = f(\tilde{x}^{i \pm s} \tilde{y}^{j+t}, x^{k+u}) = x^{i \pm s} y^{j+t} x^{k+u} = x^{i-k-u \pm s} y^{j+t} = x^{i+k+u \pm s} y^{j+t}$ (Since both u, k are 0 or 5, $x^{k+u} = x^{-k-u}$).
On the other hand, $f((a, b)) \cdot f((a', b')) = x^{i-k} y^j \cdot x^{s-u} y^t = x^{i+k \pm s \pm u} y^{j+t} = x^{i+k+u \pm s} y^{j+t}$, and like what we have shown in part b), whether we have $+s$ or $-s$ in the power of x depends on whether j is 0 or 1, but in both cases, $f((a, b) \cdot (a', b')) = f((a, b)) \cdot f((a', b'))$. Moreover, it is easy to see that for any element $x^i y^j \in D_{10}$, if $i \leq 4$, it has preimage $(x^i y^j, 1)$ otherwise its preimage is $(x^i y^j, x^5)$, thus f is surjective. Lastly, because D_{10} and $D_5 \times H$ both have 20 elements,and f is surjective, the map must also be one-to-one. Therefore f is an isomorphism of groups.
Equivalently, one can demonstrate that there is a subgroup of D_{10} that is isomorphic to D_5, namely $D* = \{1, x^2, x^4, x^6, x^8, y, x^2y, x^4y, x^6y, x^8y\}$, this subgroup has index 2 and is thus normal, and by showing that H is also normal and $H \cap D* = \{1\}$,then by a result done before, $D_{10} \simeq D_5 \times H$.

Problem 2

To show that H is dense, it suffices to show that $\forall \epsilon > 0, \exists h \in H : 0 < h < \epsilon$, because then for any arbitrarily small interval $(a, b) \in \mathbb{R}^+$, we can find $h < b - a$ and consider $\{ch\}$ where $c = \{0, 1, \dots\} \in \mathbb{Z}$. Then if c is the largest number $\in \mathbb{Z}$ such that $ch < a$, we have $a < (c + 1)h < a + h < b$, and thus any arbitrarily small interval in $\mathbb{R}^+$ contains an element in H, or H is dense.
Now observe that any element of H has the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$, and the real product of any two such elements $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2} \in H$. Thus $(\sqrt{2} - 1)^n \in H \ \forall n$. Now $|(\sqrt{2} - 1)| < 1$, thus $lim_{n\to\infty}(\sqrt{2} - 1)^n = 0$, in other words, we can always find an arbitrarily small number in H as claimed.
Another solution: By the same argument as in Artin 4.4.5, we can say that any discrete subgroup of the $\mathbb{R}^+$ is either $\{0\}$ or $\{ma | m \in \mathbb{Z}\}$, or in other words, if H is non trivial, it is generated by some minimal element a.Then both 1 and $\sqrt{2}$ are integral multiples of a. However,then let $1 = m_1 a$, $\sqrt{2} = m_2 a$, then $\frac{sqrt2}{1} = \frac{m2}{m1}$, or $\sqrt{2}$ is rational, which leads to contradiction.

Problem 3

Let the group in consideration be $\Gamma \in O$. We know that there is a homomorphism $f : O \to < \pm 1 >$ that takes an element of the group to its determinant. Now the $ker f = SO$, and when we restrict it to Γ, the kernel of the restriction is $ker f|_{res} = \Gamma \cap SO$, or the subgroup of motions in Γ. We first show that this subgroup is finite. Indeed, since the subgroup is discrete, there is a smallest element that is a rotation of smallest degree r_θ. Now for any other rotation r_ϕ, if $\phi \neq k\theta$ for some $k \in \mathbb{Z}$, then $(k-1)\theta < \phi < k\theta$. Then $k\theta - \phi < \theta$ is a an angle of rotation in Γ (by closure of addition), but is smaller than θ which contradicts our assumptions, thus all other rotations in Γ is a multiple of r_θ (essentially an euclidean algorithm argument like what we did in class). Similarly, an Euclidean argument can be used to show that $2\pi = n\theta$ for some $n \in \mathbb{Z}$. Thus, there are at most $\frac{2\pi}{\theta}$ distinct rotations, and hence $\Gamma \cap SO$ is finite. Now the image of f has order 2, thus the kernel of f has index 2 or 1 in the original group, hence Γ can not have order larger than $2 * |(\Gamma \cap SO)|$, which is finite.

Problem 4

G' acts on S by $g', s \to g's$, so let G act on S by $g, s \to \phi(g)s$ .We check that this is a group action: $(g_1 g_2)s = (\phi(g_1 g_2))s = (\phi(g_1)\phi(g_2))s = (\phi(g_1)(\phi(g_2)s) = \phi(g_1)(g_2 s) = g_1(g_2 s);\ es = \phi(e)s = e's, e'$ is the identity in G'.

However, it is not necessarily possible to go in the other direction, because ϕ is not injective and there is no natural way to define a unique preimage $g' \in G'$ for $g \in G$. It is only possible when ϕ is bijective, and we can take the inverse function ϕ^{-1} to define the action $g', s \to g's = \phi^{-1}(g)$.

Problem 5

We use the formula $|O_s| = \frac{|G|}{|G_s|}$ to check our answers:

a) The stabilizer of a vertex is the group of 2 elements: the identity and the reflection about the line through that vertex and the center of the square, we have $|O_{vertices}| = 4 = \frac{|G|}{|G_{vertex}|} = \frac{8}{2}$.

Similarly, the stabilizer of an edge, which is also of an orbit of 4 elements, consists of the identity and a reflection about the line through the midpoints of the edge and its opposite, because $\frac{|G|}{|O_{edges}|} = 2$.

b) The orbit of a diagonal consists of itself and the other diagonals, so there are 4 elements in the stabilizer, namely : the identity, rotation by π around the center of the squares, and reflections about either of the diagonals.

Problem 6

a) Now for any linear transformation $T \in GL_n(\mathbb{R})$, $T(\vec{0}) = \vec{0}, T(\vec{v}) \neq \vec{0}$, for $\vec{v} \neq \vec{0}$, hence the $O_{\vec{0}} = \{\vec{0}\}$.Any other vector $\vec{v}$ in $\mathbb{R}^n$ is the image of the basis vector e_1 under a matrix with the first column $\vec{v}$. Thus they form a single orbit, and we have $\mathbb{R}^n = \{\vec{0}\} \cup \mathbb{R}^n - \vec{0}$.

b) Now if $A \in GL_n(\mathbb{R})$ stabilizes e_1, then $Ae_1 = e_1$, but $A_e 1$ is simply the first column of A, thus we have $A = \begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ 0 & * & \dots & * \\ 0 & * & \dots & * \end{pmatrix}$. Moreover, by calculating the determinant of A in this form, we find that A has the form $A = \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \\ \vdots & & X & \\ 0 & & & \end{pmatrix}$, where $det X \neq 0$, or $X \in GL_{n-1}(\mathbb{R})$.

Problem 7

Let $g \in G$ then $gaH = aH \Leftrightarrow \exists h' \in H$ such that $ga = ah' \Leftrightarrow g = ah'a^{-1} \Leftrightarrow g \in aHa^{-1}$ Therefore the stabilizer of aH is aHa^{-1}.

Problem 8

a) $g \in G_{\phi(s)} \Leftrightarrow g(\phi(s)) = \phi(s) \Leftrightarrow \phi(gs) = \phi(s)$
Obviously, if $g \in G_s$, then $\phi(gs) = \phi(s)$, thus $g \in G_{\phi(s)}$, or i.e. $G_s \subset G_{\phi(s)}$.
b) Consider an element $s' = gs \in O_s$, then $\phi(s') = \phi(gs) = g(\phi(s)) \in O_{\phi(s)}$ Likewise, if $s' = g(\phi(s)) \in O_{\phi(s)}$, then $s' = \phi(g(s)) \in \phi(O_s)$ Thus ϕ maps $O_s \to O_{\phi(s)}$.

Problem 9

Note: We can not assume that G is finite. a)We show first that if $K \subset H \subset G$, then $[G : K] = [G : H][H : K]$, essentially by showing that we the cosets of K under G can be counted as the cosets of K under H in the cosets of H under G Now as G acts on H by permuting the cosets of H, each coset has representative gH, for some $g \ni nH$ There are $[G : H]$ such cosets.
In the coset H, H acts on itself, and permutes the cosets of the subgroup K, each coset has a representative hK. In any other coset gH, H also acts on the coset, and permutes K cosets within that coset, each such K coset has the representative ghK. Now since the cosets partition the set being acted upon, all such K cosets are disjoint and indeed partition G. Furthermore, let G act on these K coset, we see that $g(K) = (g'h)K = g'(hK)$, where $g \in g'H$ thus G permutes the K cosets as well. In conclusion, we can see the cosets of K in each coset gH as being acted upon by G, and there are exactly $[G : H] * [H : K]$ such cosets. Thus, the number of cosets of K in G is $[G : K] = [G : H][H : K]$. Now, back to our main question, $H \cap K$ is a subgroup of H, thus applying what we just proved, $[G : H \cap K] = [G : H][H : H \cap K]$. From Artin 7 5, $[H : H \cap K] \leq [G : K]$, thus all factors are finite, and hence $[G : H \cap K]$ is also finite.
b) In $G = S_3$, take $H = \{(12), e\}$, $K = \{(23), e\}$. Then $H \cap K = \{e\}$ thus $[H : H \cap K] = 2$, but $[G : K] = \frac{|G|}{|K|} = 3$, thus $[H : H \cap K] \not| [G : K]$.
General note: The most important issue in this problem is to differentiate the action of G and the action of H. In many write-ups, you showed very rigorously that the K cosets are disjoint and that their number adds up to $[G : H][H : K]$, but you need to show that G acts on K and permutes the K-cosets to justify using $[G : K]$ as the number of K cosets in G.

Problem 10

(a) $\varphi(g_1 \cdot g_2) = m_{g_1 g_2}, \varphi(g_1) \cdot \varphi(g_2) = m_{g_1} m_{g_2}$, apply these maps to any element s in $S : m_{g_1 g_2}(s) = (g_1 g_2)(s) = g_1(g_2 s) = m_{g_1} m_{g_2}(s)$. Also, $\varphi(e)(s) = m_e s = es$, thus φ is indeed a homomorphism.
(b) Let the operation be $G \times S \to S, (g, s) \to gs = \varphi(g)(s)$. Then under this action $(e, s) \to \varphi(e)s = e(s) = s$ (the action of the identity permutation), $g_1 g_2, s \to (g_1 g_2)s = \varphi(g_1 g_2)(s) = (\varphi(g_1)\varphi(g_2))(s) = \varphi(g_1)(\varphi(g_2)(s)) = g_1(g_2 s) \leftarrow g_1, g_2 s$.
(c) Note that when we have to prove that a map between functions is the identity, we show that the original function and the image functions agree on all points in the domain of the function. Similarly, when we need to show two actions are the same, show that they acts the same on each element of the set S
Consider first $\psi \circ \psi'$, which is a map between homomorphisms φ of $G \to$ Perm(S). According to part b), $\psi'(\varphi)$= f, where f is an action of G on S, as defined in part (b), $f : g, s \to gs = \varphi(g)(s)$. Now, applying ψ to f, according to part a), we have $\psi(f) = \phi$, where $\phi(g)$ is a permutation of S such that $\phi(g)(s) = m_g(s)$(left multiplication of g to s under f) $= gs$(action of g on s under f) $= \varphi(g)(s)$. Thus for each $g, \phi(g)$ and $\varphi(g)$ are the same permutation of S, therefore $\psi(f) = \phi = \varphi$, or $\psi \circ \psi' =$ id.

Consider $\psi' \circ \psi$, which is a map between actions $f : G \times S \rightarrow S$. Let $\psi' \circ \psi(f) = \psi'(\varphi)$, where $\varphi : G \rightarrow$ Perm(S), each $\varphi(g)$ is a permutation of S such that $\varphi(g)(s) = m_g(s) = gs$(under f). Then by part (b), $\psi'(\varphi)$ = an action f': $G \times S \rightarrow S$ for each g, f' acts such that $g, s \rightarrow gs$ (under f') $= \varphi(g)(s) = m_g s$ (this is multiplication of g to s under f by part a) $= gs$(under f). Thus the action f' for each g = action of f for each g, and $\psi' \circ \psi$ = id, or there is indeed a bijection.

General comments: This problem introduces a useful technique for proving equivalence or congruence: show that there is a pair of maps whose compositions are identity maps. Also, while this exercise might seem like moving symbols around, it really is about the distinction between actions of f, of f', or action of a permutation $\varphi(g)$ on the set S. A common mistake is using conjugation in part b) when defining the action. Conjugation between element in S and in G are not defined, until we define multiplication of G with S, which is exactly what we need to do. We just need to define how that multiplication is determined in term of φ to define the action.